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Secondary bifurcation and change of type for three-dimensional standing waves in finite depth

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The nonlinear periodic free oscillations of irrotational surface waves in a three-dimensional basin with a rectangular cross-section and finite depth are considered. A previous work by Verma & Keller (1962) has analysed the case when the linear natural frequencies are non-commensurate. For particular values of the parameters, however, strong internal resonance occurs (two natural frequencies are equal). Instead of the usual loss of stability and exchange of energy, it is found that the double eigenvalue generates a higher multiplicity of periodic solutions. Eight solution branches are found to be emitted by the double eigenvalues. It is also shown that perturbing the double eigenvalue results in a secondary bifurcation of periodic solutions. The direction of the branches for the multiple and secondary bifurcation changes with the depth. Finally it is shown that the formal solutions obtained are not uniformly valid and an additional expansion in the Boussinesq regime shows that the wave field changes type. One of the solutions in this regime is a field of three-dimensional cnoidal standing waves.

1. Introduction

In this paper the nonlinear periodic free oscillations of irrotational water waves in a three-dimensional basin with a rectangular cross-section are considered. A linearized analysis of this problem shows that there is a countably infinite set of linear eigenvalues or natural frequencies with associated eigenfunctions. The eigenvalues are strictly real (strictly imaginary in the usual terminology) owing to the Hamiltonian nature of irrotational water waves. Therefore it is convenient to use the terminology and approach of equilibrium bifurcation theory, treating the natural frequency as the bifurcation parameter. Following this approach each of the linear eigenvalues is considered to be a point of bifurcation for the finite-amplitude solution. The solutions at the bifurcation points are periodic solutions, and finite-amplitude branches of periodic solutions emitted by these points are sought.

Verma & Keller (1962) have used perturbation methods to determine the natural frequency and eigenfunctions for finite-amplitude periodic waves emitted by simple eigenvalues. However, in addition to the set of simple eigenvalues it is easily shown that there is a countably infinite set of double eigenvalues, triple eigenvalues, and quadruple eigenvalues. The bifurcation points with higher-dimensional null spaces are interesting because they generate a higher multiplicity of periodic solutions; moreover, perturbation of the degenerate points may result in secondary branching at finite amplitude of new periodic solutions.

The appearance of double eigenvalues or eigenvalues that are commensurate in dynamical systems is often referred to as internal resonance. A typical example is the spherical pendulum analysed by Miles (1962) which has two degrees of freedom with equal linear natural frequencies. Miles's analysis of this problem shows that internal resonance results in a loss of stability and exchange of energy between the two modes. The exchange of energy breaks the periodic solutions and produces modulated periodic solutions. In this context Miles (1984*a*, *b*) has analysed irrotational water waves in a basin with circular cross-section when a double eigenvalue occurs with a pair of linearly independent eigenfunctions. He showed that the flow of the amplitudes of the eigenfunctions corresponding to a double natural frequency is analogous to internal resonance of the spherical pendulum. Periodic solutions are found for specific initial conditions and instabilities leading to a slow exchange of energy and modulated amplitudes are found.

It is shown herein that the qualitative results for waves in a rectangular basin under similar circumstances are quite different. An internal resonance in this case generates a higher multiplicity of periodic solutions which are stable, and apparently there is no exchange of energy between the two eigenfunctions. It was mentioned above that Miles showed that the internal resonance in the circular basin is analogous to the spherical pendulum. It is useful to construct a pendulum analogy for the waves in a basin of rectangular cross-section to help illustrate why the solutions in a circular and rectangular cross-section under similar circumstances are qualitatively different.

Consider a right-handed three-dimensional coordinate system with the y-axis directed upwards and the x-axis directed to the right with a pendulum of mass m_1 and length l_1 suspended from the origin with its motion restricted to the (x, y)-plane. Suspended from the pendulum mass m_1 is a second pendulum of mass m_2 and length l_2 whose motion is restricted to the (y, z)-plane. The natural frequencies of the two pendulums are equal when $l_1 = l_2$. Under this circumstance one would expect the usual exchange of energy between the modes which occurs in other systems with internal resonance. However, based on the fact that the planes of motion are orthogonal, it may be shown that a higher multiplicity of periodic solutions are generated but they are stable and do not exchange energy. The analysis for this two-degree-of-freedom orthogonal planar pendulum will not be carried out here, but the result is equivariant to that shown in the results to follow. This pendulum model is analogous to what happens in the sloshing of fluid in a basin with a square cross-section when a double eigenvalue with a pair of orthogonal eigenfunctions occurs. This may be contrasted with the analysis of Miles on the basin with circular cross-section. There the symmetry of the circle plays an important role and the analogy is with the spherical pendulum. Because the two-degree-of-freedom spherical pendulum is less constrained than the two-degree-of-freedom orthogonal planar pendulum mentioned above, the class of solutions is larger, less stable, and even chaotic (Miles 1984*b*).

For the mathematical model a three-dimensional cylinder of rectangular cross-section partially filled with fluid is considered with the origin of the coordinate system at the still water level. The y-axis is normal to the still fluid surface and passes through the centre of the cross-section. Owing to the assumption of irrotationality the dependent variables are reduced to the velocity potential $\phi(x, y, z, t)$ and the wave height $\eta(x, z, t)$. In dimensionless form the governing equation and boundary conditions are

$$\frac{\partial \phi}{\partial x^2}+\frac{\partial \phi}{\partial y^2}+\xi^2\frac{\partial \phi}{\partial z^2} = 0, \tag{1.1}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on solid boundary,} \tag{1.2}$$

and, on $y = \epsilon\eta(x, z, t)$,

$$\omega\frac{\partial\eta}{\partial t}+\epsilon\frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x}+\epsilon\xi^2\frac{\partial\eta}{\partial z}\frac{\partial\phi}{\partial z}-\frac{\partial\phi}{\partial y} = 0, \tag{1.3}$$

$$\omega\frac{\partial\phi}{\partial t}+\frac{1}{2}\epsilon\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\xi^2\left(\frac{\partial\phi}{\partial z}\right)^2\right]+\eta = 0, \tag{1.4}$$

where $\xi = a/b$, $2a$ is the vessel length in the x-direction, $2b$ is the vessel length in the z-direction; $\delta = h/2a$, where h is the vessel still water depth; $\epsilon = H/h$ where H is a measure of the wave height.

The linearized problem for (1.1)–(1.4) has eigenvalues (bifurcation points, linear natural frequencies)

$$\sigma_0 = (\lambda_{mn}\tanh\lambda_{mn}\delta)^{\frac{1}{2}}, \tag{1.5}$$

where

$$\lambda_{mn} = \pi(m^2+\xi^2n^2)^{\frac{1}{2}}. \tag{1.6}$$

The solutions for finite but small amplitude emitted by the simple linear eigenvalues have been found by Verma & Keller (1962) using a formal perturbation expansion. However, at particular combinations of (m, n) and ξ there are multiple eigenvalues. In §2 it is found that a set of eight solution branches are emitted by the double eigenvalues. A formal perturbation expansion analogous to the approach of Verma & Keller is used.

By varying an auxiliary parameter (in this case ξ) away from critical values the double eigenvalues are 'split' into two simple eigenvalues. In an interesting discovery Bauer, Keller & Reiss (1975) observed that a secondary bifurcation may be generated when a double eigenvalue is split. At the double eigenvalue four branches are emitted in the half-plane $\epsilon > 0$ and the splitting process breaks this into two primary branches. The other two branches originating at the double eigenvalue slowly depart by creeping up a primary branch as ξ is varied away from a critical value. Bauer *et al.* developed a perturbation method to analyse this phenomenon and it has subsequently been applied to many equilibrium problems. For example Matkowsky, Putnick & Reiss (1980) have found secondary bifurcation in the buckling of rectangular plates, Kriegsmann & Reiss (1978) have found secondary bifurcation of magnetohydrodynamic equilibria, and the analysis has been extended to the bifurcation from triple eigenvalues, which results in secondary and tertiary bifurcation, by Reiss (1983).

In §3 this theory is applied to the splitting of double natural frequencies and it is shown that the splitting generates a secondary bifurcation of periodic solutions at finite amplitude. The secondary bifurcation points for a perturbed square cross-section are found as functions of δ and the mode numbers, and expressions for the solutions along the secondary branches are derived. It is found that the jump to a secondary branch produces interesting irregular wave forms.

It is usual in the theory of water waves to consider three regions in the amplitude–depth parameter space: $\epsilon \ll \delta^2$ is the deep-water or Stokes regime; $\epsilon \gg \delta^2$ is the shallow-water regime where the governing equations are analogous to those of a compressible gas and give rise to hydraulic jumps; and $\epsilon = O(\delta^2)$ is the Boussinesq regime where the amplitude and dispersion are in balance (the ratio ϵ/δ^2 is often referred to as the Ursell number). The perturbation expansions of Verma & Keller, the finite-amplitude solutions found in §2 emitted by double eigenvalues, and the secondary bifurcation phenomena elucidated in §3 are valid in the Stokes regime only. It is shown that when $\epsilon = O(\delta^2)$ the higher-order terms are no longer of higher order and the expansions break down.

In §4 a separate analysis is performed in the Boussinesq regime by taking the small parameters ϵ and δ^2 to be of equal order. As a first approximation weakly three-dimensional waves are considered. This is done by looking in the region of the (ξ, δ, ϵ)-parameter space where the triple balance

$$\xi^2 = O(\epsilon),$$
$$\delta^2 = O(\epsilon)$$

holds. This analysis results in a field of standing K–P waves, a set of two non-interacting (to first order) solutions of the K–P equation (Kadomtsev & Petviashvili 1970). The K–P equation, which is rich in solutions, has been studied in some detail by Dubrovin (1981) and Segur & Finkel (1985). As these solutions are qualitatively different from those found in the Stokes regime we say that the wave field changes type as the amplitude reaches $\epsilon = O(\delta^2)$. Since the analysis is for a particular region of the (ϵ, δ)-plane the solutions are not contiguous to those derived in §§2 and 3.

The results with the K–P equations are for weakly three-dimensional waves. A further analysis is performed in §4 with this assumption relaxed. This analysis results in a wave equation to leading order. The solvability condition at the next order results in a functional differential equation for the leading-order term. A complete solution is not found, but it is shown by substitution that one solution is a set of four oblique, travelling cnoidal waves, which combine to form a three-dimensional standing wave. Distributions of the wave height are shown for these waves. It is expected that this equation will yield other interesting possibilities.

2. Primary bifurcation when $\epsilon < O(\delta^2)$

Linearizing the set of equations about the still water level results in a linear problem that has eigenvalues (linear natural frequencies)

$$\sigma_0 = (\lambda_{mn} \tanh \lambda_{mn} \delta)^{\frac{1}{2}} \tag{2.1}$$

and eigenfunctions

$$\eta_1 = \cos \alpha_m \bar{x} \cos \beta_n \bar{z} \sin t, \tag{2.2}$$

$$\phi_1 = \sigma_0^{-1} \frac{\cosh \lambda_{mn}(y+\delta)}{\cosh \lambda_{mn} \delta} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} \cos t, \tag{2.3}$$

where $\lambda_{mn} = (\alpha_m^2 + \xi^2 \beta_n^2)^{\frac{1}{2}}$, $\alpha_m = m\pi$, $\beta_n = n\pi$, $\bar{x} = x + \frac{1}{2}$, $\bar{z} = z + \frac{1}{2}$, and m, n are the mode numbers in the (x, z)-directions.

The solutions that bifurcate from the linear eigenvalues (2.1) were first found by Verma & Keller (1962) using a perturbation expansion in the amplitude. They found that the natural frequency for the three-dimensional standing wave has the following form as $\epsilon \to 0$:

$$\omega = \sigma_0 \left(1 + \epsilon^2 \frac{\sigma_2}{\sigma_0} + \dots\right), \tag{2.4}$$

where

$$\frac{\sigma_2}{\sigma_0} = \frac{1}{256}\left\{9\frac{\lambda_{mn}^6}{\sigma_0^8} - 24\frac{(\alpha_m^4 + \xi^4\beta_n^4)}{\sigma_0^4} + 5\lambda_{mn}^2 - 46\sigma_0^4\right\} - \frac{(3\sigma_0^4 + \lambda_{mn}^2 - 4\alpha_m^2)^2}{64\sigma_0^2(\alpha_m \tanh 2\alpha_m \delta - 2\sigma_0^2)} - \frac{(3\sigma_0^4 + \lambda_{mn}^2 - 4\xi^2\beta_n^2)^2}{64\sigma_0^2(\beta_n \xi \tanh 2\beta_n \xi\delta - 2\sigma_0^2)}. \tag{2.5}$$

Taking $\alpha_m = \beta_n = 1$ and $\xi = 1/L$ this agrees with the expression in the paper by Verma & Keller. The bifurcation for the frequency is subcritical or supercritical depending on the value of δ. Here a subcritical (supercritical) branch means the natural frequency decreases (increases) as the amplitude increases. As $\delta \rightarrow \infty$ (deep water) the bifurcation is subcritical and as $\delta \rightarrow 0$ (shallow water) it is supercritical. Figure 2 of the paper by Verma & Keller shows, for the first mode, that as the vessel cross-section departs from being a square the critical depth δ^* (the point where the bifurcation changes from sub- to supercritical) increases. The same phenomenon occurs for the higher modes as well, with the critical depth being smaller as the mode number increases (for fixed ξ). Consequently when a lower mode is supercritical it may be that a higher mode, with all other parameters being equal, may bifurcate subcritically, which suggests the possibility of an intersection of the branches at finite amplitude.

It can be shown however that the asymptotic solution obtained by Verma & Keller is not uniformly valid. Taking the limit as $\delta \rightarrow 0$ it is found that

$$\frac{\sigma^2}{\sigma_0} \rightarrow \frac{k}{\lambda_{mn}^2 \delta^4},$$

where k is a constant of $O(1)$. Therefore the solution is valid for $\epsilon < O(\delta^2)$ only. In §4 the equations will be reanalysed for the region $\epsilon = O(\delta^2)$, and it will be shown that the solutions change type in this region.

Inspection of (2.1) also shows that the linearized problem has double eigenvalues at particular combinations of α_m, β_n, and ξ. When $\xi = 1$ every pair (α_m, β_n) such that $m \neq n$ is a double eigenvalue. Other examples are $\xi = \frac{1}{2}$ which results in $\lambda_{1,4} = \lambda_{2,2}$, and $\xi = 1$ which results in the triple eigenvalue $\lambda_{1,7} = \lambda_{5,5} = \lambda_{7,1}$. In fact every rational ξ will have an infinite set of multiple eigenvalues. It will now be shown that the double eigenvalues emit multiple branches of solutions. For brevity the problem of a square vessel ($\xi = 1$) will be considered; it is expected that the analysis at other double eigenvalues will result in a similar conclusion. When $\xi = 1$ the bifurcation points occur at (2.1) with

$$\lambda_{mn} = (\alpha_m^2 + \beta_n^2)^{\frac{1}{2}}. \tag{2.6}$$

Therefore any pair (m, n) such that $m \neq n$ will result in a double eigenvalue. The analysis proceeds by formally expanding the frequency, potential, and wave height in a regular perturbation series. The leading term in the frequency expansion is given by (2.1) with (2.6) and the leading term for each of the dependent variables is

$$\eta_1 = [A_{11} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} + A_{12} \cos \beta_n \bar{x} \cos \alpha_m \bar{z}] \sin t, \tag{2.7a}$$

$$\phi_1 = \frac{1}{\sigma_0}[A_{11} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} + A_{12} \cos \beta_n \bar{x} \cos \alpha_m \bar{z}] \frac{\cosh \lambda_{mn}(y+\delta)}{\cosh \lambda_{mn} \delta} \cos t. \tag{2.7b}$$

A normalization for the coefficients is chosen such that

$$A_{11}^2 + A_{12}^2 = 1. \tag{2.8}$$

The relative magnitudes of A_{11} and A_{22} are determined at higher order.

Proceeding in the usual way results in the following set of bifurcation equations after application of the double solvability condition at the third order:

$$[a_1 A_{11}^2 + a_2 A_{12}^2 + 2\sigma_2] A_{11} = 0, \tag{2.9a}$$

$$[a_2 A_{11}^2 + a_1 A_{12}^2 + 2\sigma_2] A_{12} = 0, \tag{2.9b}$$

which along with (2.8) form a set of three equations for the three unknowns: σ_2, A_{11}, A_{12}. The coefficients a_1 and a_2 are given by

$$a_1 = \frac{\sigma_0}{128}\left[-9\frac{\lambda_{mn}^6}{\sigma_0^8}+24\frac{(\alpha_m^4+\beta_n^4)}{\sigma_0^4}-5\lambda_{mn}^2+46\sigma_0^4\right] + \frac{(3\sigma_0^4-3\alpha_m^2+\beta_n^2)^2}{32\sigma_0[\alpha_m \tanh 2\alpha_m\delta-2\sigma_0^2]}+\frac{(3\sigma_0^4-3\beta_n^2+\alpha_m^2)^2}{32\sigma_0[\beta_n \tanh 2\beta_n\delta-2\sigma_0^2]}, \tag{2.10}$$

$$a_2 = \frac{(3\sigma_0^4-\lambda_{mn}^2-4\alpha_m\beta_n)^2}{16\sigma_0[\sqrt{2}(\alpha_m+\beta_n)\tanh[\sqrt{2}(\alpha_m+\beta_n)\delta]-4\sigma_0^2]} + \frac{(3\sigma_0^4-\lambda_{mn}^2+4\alpha_m\beta_n)^2}{16\sigma_0[\sqrt{2}(\alpha_m-\beta_n)\tanh[\sqrt{2}(\alpha_m-\beta_n)\delta]-4\sigma_0^2]} + \frac{(3\sigma_0^4-\lambda_{mn}^2)^2}{8\sigma_0[\sqrt{2}\lambda_{mn}\tanh[\sqrt{2}\lambda_{mn}\delta]-4\sigma_0^2]} + \frac{\sigma_0}{16}\left\{11\sigma_0^4+3\frac{\lambda_{mn}^4}{\sigma_0^4}-2\lambda_{mn}^2-3\frac{(\alpha_m^4+\beta_n^4)}{\sigma_0^4}\right\}. \tag{2.11}$$

The three equations (2.8) and (2.9) have the following set of eight solutions:

$$\left.\begin{array}{l}\text{Pure No. 1:}\quad A_{11}=\pm 1,\quad A_{12}=0,\quad \sigma_2^{\mathrm{p}}=-\tfrac{1}{2}a_1,\\ \text{Pure No. 2:}\quad A_{11}=0,\quad A_{12}=\pm 1,\quad \sigma_2^{\mathrm{p}}=-\tfrac{1}{2}a_1,\\ \text{Mixed:}\quad A_{11}=\pm\dfrac{1}{\sqrt{2}},\quad A_{12}=\pm\dfrac{1}{\sqrt{2}},\quad \sigma_2^{\mathrm{m}}=-\tfrac{1}{4}(a_1+a_2).\end{array}\right\} \tag{2.12}$$

The pure modes correspond to the modes of the simple eigenvalues that coalesce to form the double point. They share the same natural frequency and are spatially (horizontally) symmetric. When the relevant parameters are substituted the amplitude correction to the natural frequency $\sigma_2^{\mathrm{p}} = -\frac{1}{2}a_1$ agrees with the correction found for the simple eigenvalues ((2.5) with $\xi = 1$). The other four solutions are mixed modes. The leading terms are proportional to the sum or difference of the two pure eigenfunctions. The amplitude correction of the frequency for the mixed-mode solutions σ_2^{m} differs from that for the pure modes by an amount $\frac{1}{4}(a_1-a_2)$. The bifurcation for either the pure or mixed modes will be sub- or supercritical depending on the value of δ.

Figures 1 (a, b, c, d, e) are bifurcation diagrams for the natural frequency for the multiple eigenvalues occurring in a square cross-section with mode numbers $m = 1$ and $n = 2$. The five frames show the effect of the parameter δ on the behaviour of the solutions. Figure 1 (a) corresponds to infinite depth and agrees with the result in Bridges (1987), where a more complete study of the infinite-depth case is given. The mixed branch apparently has a higher natural frequency for the range of amplitudes considered and for all values of the depth. The remainder of the figure 1 set correspond to decreasing values of δ. As δ decreases the branches all shift to the right and eventually (at the critical depth δ^*) shift from sub- to supercritical. At $\delta = 0.1275$ there is the interesting property that the pure branch bifurcates subcritically and the mixed branch bifurcates supercritically.

In Bridges (1987) distributions of the wave height for the pure branches and the mixed-mode branches for $\delta \rightarrow \infty$ are shown. It is expected that, qualitatively, the distributions for finite δ (and suitably restricted ϵ) will be similar.

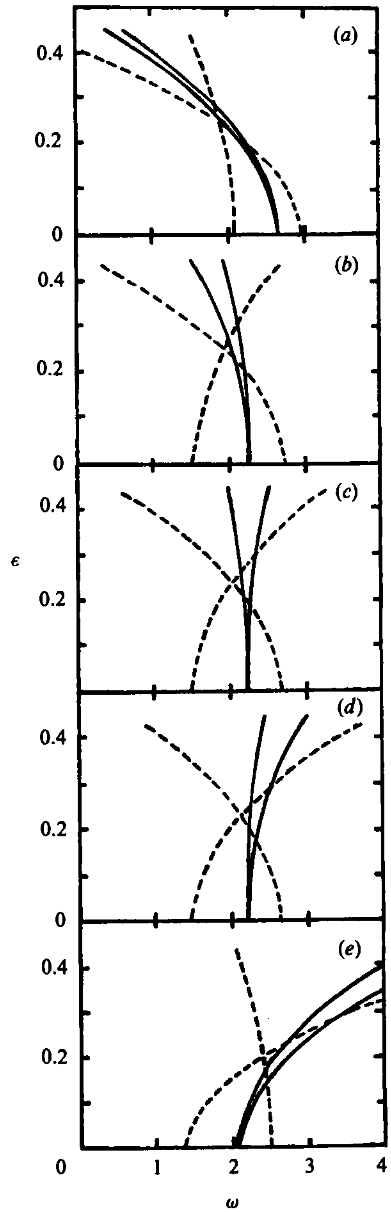

FIGURE 1. Effect of δ on the bifurcation from the double eigenvalues for $\xi = 1$ and (m, n) ranging over 1, 2. The dashed lines correspond to the simple modes $\sigma_{1,1}$ (left branch) and $\sigma_{2,2}$ (right branch) and the solid lines are for pure (left branch) and mixed (right branch) modes emitted by the double point $\sigma_{1,2}$, $\sigma_{2,1}$. (*a*) $\delta = \infty$; (*b*) 0.14; (*c*) 0.1275; (*d*) 0.12; (*e*) 0.10.

In contrast to the usual result for internal resonance where loss of stability and energy exchange occurs between the resonant modes it has been shown that, under the special circumstances inherent in this problem, a higher multiplicity of stable periodic solutions are found. Stability may be shown using a two-timing method. Define the slow scale $T = \epsilon^2 t$ and use the more general linear solution, which allows for a temporal phase,

$$\eta_1 = f_{11}(t, T) \cos \alpha_m \bar{x} \cos \beta_n \bar{z} + f_{12}(t, T) \cos \beta_n \bar{x} \cos \alpha_m \bar{z}, \tag{2.13}$$

where

$$f_{1k} = A_{1k}(T)\,\mathrm{e}^{\mathrm{i}t} + \text{c.c.} \quad \text{for } k = 1, 2, \tag{2.14}$$

where c.c. denotes complex conjugate, and the A_{1k} are complex amplitudes. Carrying this to third order in the usual way results in a modified set of bifurcation equations

$$\mathrm{i}\gamma \frac{\mathrm{d}A_{11}}{\mathrm{d}T} = 2\sigma_2 A_{11} + a_1|A_{11}|^2 A_{11} + a_2|A_{12}|^2 A_{11}, \tag{2.15a}$$

$$\mathrm{i}\gamma \frac{\mathrm{d}A_{12}}{\mathrm{d}T} = 2\sigma_2 A_{12} + a_2|A_{11}|^2 A_{12} + a_1|A_{12}|^2 A_{12}, \tag{2.15b}$$

where γ is a real number. Taking $A_{1k} = R_{1k}(T)\,\mathrm{e}^{\mathrm{i}\psi_{1k}(T)}$ for $k = 1, 2$ it is clear that $\mathrm{d}R_{1k}/\mathrm{d}T = 0$ for $k = 1, 2$ implying stability of each branch of free oscillations.

3. Secondary bifurcation when $\epsilon < O(\delta^2)$

In this section new secondary branches of periodic solutions, which intersect a primary branch emitted by a simple eigenvalue, at finite amplitude, are sought. Therefore a perturbation

$$\phi = \epsilon' \Phi + \phi', \tag{3.1a}$$

$$\eta = \epsilon' \Upsilon + \eta' \tag{3.1b}$$

is added to the known primary-branch solutions (Φ, Υ, ω) found by Verma & Keller. These expressions are substituted into the governing equations and boundary conditions, which are then linearized about the known primary-branch solutions. The linear problem to be solved for the points, if they exist, of secondary bifurcation is

$$\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \xi^2 \frac{\partial^2 \phi'}{\partial z^2} = 0, \tag{3.2}$$

and on $y = \epsilon' \Upsilon(x, z, t)$,

$$\omega \frac{\partial \eta'}{\partial t} + \epsilon' \frac{\partial \Phi}{\partial x}\frac{\partial \eta'}{\partial x} + \epsilon' \frac{\partial \Upsilon}{\partial x}\frac{\partial \phi'}{\partial x} + \epsilon' \xi^2 \left[\frac{\partial \Phi}{\partial z}\frac{\partial \eta'}{\partial z} + \frac{\partial \Upsilon}{\partial z}\frac{\partial \phi'}{\partial z}\right] - \frac{\partial \phi'}{\partial y} = 0, \tag{3.3}$$

and

$$\omega \frac{\partial \phi'}{\partial t} + \epsilon' \left[\frac{\partial \Phi}{\partial x}\frac{\partial \phi'}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \phi'}{\partial y} + \frac{\partial \Phi}{\partial z}\frac{\partial \phi'}{\partial z}\right] + \eta' = 0, \tag{3.4}$$

and in addition the normal derivative is required to vanish on the solid boundary. This is a linear differential eigenvalue problem with known non-constant coefficients. However, the specific value of ϵ' where the secondary bifurcation takes place is sought. Therefore ϵ' is the eigenvalue. Since ϵ' appears nonlinearly it is a problem nonlinear in the eigenvalue parameter and there is the further complication that ϵ' is responsible for the size of the domain. The qualitative shape of the domain is known since $\Upsilon(x, z, t)$ is a known function, but the precise multiple of $\Upsilon(x, z, t)$ is the unknown eigenvalue. This is to be contrasted with the original eigenvalue problem in §2 where ω was the eigenvalue, ϵ' was a variable parameter, and the shape of the free surface (and hence the domain) was an unknown function.

To solve this eigenvalue problem, the conjecture of Bauer *et al.* (1975), that a secondary bifurcation may occur in the neighbourhood of a multiple eigenvalue, is used. For brevity, the analysis is undertaken in the neighbourhood of $\xi = 1$. It is expected that a similar analysis will hold in the neighbourhood of other values of ξ at which double eigenvalues occur.

It was shown in §2 that for $\xi = 1$ there is a double eigenvalue for every pair (m, n) such that $m \neq n$. At the double eigenvalue the bifurcation points are given by (2.1) with (2.6). In the neighbourhood of $\xi = 1$ this double eigenvalue splits into two primary branches emitted by the bifurcation points.

$$\sigma_{m,n}(\delta, \xi) = \{\pi(m^2+\xi^2 n^2)^{\frac{1}{2}} \tanh [\pi\delta(m^2+\xi^2 n^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}, \tag{3.5}$$

$$\sigma_{n,m}(\delta, \xi) = \{\pi(n^2+\xi^2 m^2)^{\frac{1}{2}} \tanh [\pi\delta(n^2+\xi^2 m^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}, \tag{3.6}$$

where $\sigma_{m,n}(\delta, 1) = \sigma_{n,m}(\delta, 1)$. A measure of the neighbourhood of $\xi = 1$ is given by the small parameter μ defined by

$$\xi = 1+\tfrac{1}{2}\tau\mu^2 \quad \text{where } \tau = \operatorname{sgn}(\xi-1) = \pm 1. \tag{3.7}$$

Following the conjecture of Bauer *et al.* that the secondary bifurcation disappears at the double eigenvalue, the point ϵ' on the primary branches where the secondary bifurcation will take place is expressed as

$$\epsilon'(\mu) = b_0\mu + b_1\mu^2 + \dots. \tag{3.8}$$

The solutions previously obtained by Verma & Keller for the primary branches emitted by simple eigenvalues are recast, using (3.7)–(3.9), as expansions in the small parameter μ,

$$\Phi = \psi_0 + \mu\psi_1 + \dots, \tag{3.9a}$$

$$\Upsilon = \Upsilon_0 + \mu\Upsilon_1 + \dots, \tag{3.9b}$$

$$\omega = \omega_0 + \mu^2\omega_2 + \dots. \tag{3.10}$$

A separate analysis is undertaken for each of the primary branches σ_{mn} and σ_{nm}. The necessary details for the analysis along the σ_{mn} branch will be given and the result only will be stated for the σ_{nm} branch.

Substitute the expressions (3.7)–(3.10) into the linear eigenvalue problem and postulate that

$$\phi' = \mu\phi_1' + \mu^2\phi_2' + \dots, \tag{3.11a}$$

$$\eta' = \mu\eta_1' + \mu^2\eta_2' + \dots. \tag{3.11b}$$

Expanding the free-surface boundary conditions in a Taylor series, and equating terms proportional to like powers of μ to zero results in a sequence of boundary-value problems. The fact that ω_0 is a double eigenvalue results in the leading term in the set

$$\omega_0 = (\lambda_0 \tanh \lambda_0 \delta)^{\frac{1}{2}}, \tag{3.12}$$

where

$$\lambda_0 = \pi(m^2+n^2)^{\frac{1}{2}}, \tag{3.13}$$

$$\eta_1'(x, z, t) = [A_{11} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} + A_{12} \cos \beta_n \bar{x} \cos \alpha_m \bar{z}] \sin t, \tag{3.14}$$

$$\phi_1'(x, y, z, t) = \frac{1}{\omega_0} \frac{\cosh \lambda_0(y+\delta)}{\cosh \lambda_0 \delta} [A_{11} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} + A_{12} \cos \beta_n \bar{x} \cos \alpha_m \bar{z}] \cos t, \tag{3.15}$$

and the normalization is taken to be

$$A_{11}^2 + A_{12}^2 = 1. \tag{3.16}$$

The problem is carried in the usual way to higher order. At third order application of the double solvability condition results in the equations

$$b_0^2 A_{11} = 0, \tag{3.17}$$

$$\left[\frac{\tau}{\omega_0^2}(\alpha_m^2-\beta_n^2)\left\{\frac{\omega_0^2}{2\lambda_0^2}+(\lambda_0^2-\omega_0^4)\frac{\delta}{2\lambda_0^2}\right\}+a_3 b_0^2\right]A_{12} = 0. \tag{3.18}$$

which with (3.16) form a set of three equations for the three unknowns b_0, A_{11}, and A_{12}. The term a_3 is given by

$$\begin{aligned} a_3(\delta) = {} & \frac{1}{128}\left\{27\lambda_0^2+14\omega_0^4-24\frac{\lambda_0^4}{\omega_0^4}-9\frac{\lambda_0^6}{\omega_0^8}\right\}+\frac{3(\alpha_m^4+\beta_n^4)}{8\omega_0^4} \\ & +\frac{(3\omega_0^4+\lambda_0^2-4\alpha_m^2)^2}{32\omega_0^2(\alpha_m\tanh 2\alpha_m\delta-2\omega_0^2)}+\frac{(3\omega_0^4+\lambda_0^2-4\beta_n^2)^2}{32\omega_0^2(\beta_n\tanh 2\beta_n\delta-2\omega_0^2)} \\ & -\frac{[\omega_0^4-(\alpha_m-\beta_n)^2][4\omega_0^4-\lambda_0^2+8\alpha_m\beta_n]}{16\omega_0^2\{-4\omega_0^2+\surd 2(\alpha_m-\beta_n)\tanh[\surd 2(\alpha_m-\beta_n)\delta]\}} \\ & -\frac{[\omega_0^4-(\alpha_m+\beta_n)^2][4\omega_0^4-\lambda_0^2-8\alpha_m\beta_n]}{16\omega_0^2\{-4\omega_0^2+\surd 2(\alpha_m+\beta_n)\tanh[\surd 2(\alpha_m+\beta_n)\delta]\}} \\ & -\frac{[\omega_0^4-\lambda_0^2][4\omega_0^4-\lambda_0^2]}{8\omega_0^2\{-4\omega_0^2+\surd 2\lambda_0\tanh[\surd 2\lambda_0\delta]\}}. \end{aligned} \tag{3.19}$$

For sufficiently large δ it has been shown by numerical evaluation for m, n ranging over 1 to 10 that the expression for a_3 is positive. In the limit as $\delta \rightarrow 0$ however $a_3 \rightarrow -\frac{9}{128}\lambda_0^2\delta^{-4}$. Therefore as $\delta \rightarrow 0$, a_3 will eventually become negative. For example when $(m, n) = (1, 2)$, a_3 changes sign when $\delta \sim 0.075$. When a_3 changes sign this means that the secondary bifurcation will 'jump' from one branch to another, which would be a discontinuous phenomenon. It is more likely that this behaviour is a ramification of the non-uniformity in δ of the solution.

The solutions to (3.16)–(3.18) are

Case I: $$b_0 = 0, \quad A_{11} = 1, \quad A_{12} = 0; \tag{3.20a}$$

Case II: $$b_0 = \pm\left[-\frac{(\alpha_m^2-\beta_n^2)\tau}{2a_3\omega_0^4}\left\{\frac{\omega_0^2}{2\lambda_0^2}+(\lambda_0^2-\omega_0^4)\frac{\delta}{2\lambda_0^2}\right\}\right]^{\frac{1}{2}}, \quad A_{11} = 0, \quad A_{12} = 1. \tag{3.20b}$$

The solution in (3.20*a*) shows that the basic solution bifurcates from the primary branch. In (3.20*b*) the solution for the secondary bifurcation on branch σ_{mn} is given. The $\pm$ sign shows that the bifurcation takes place in both the upper and lower ϵ half-planes. The jump to the $A_{12} \neq 0$ solution is often referred to as mode jumping because the solution acquired on the secondary branch is qualitatively different from that on the primary branch. The radical in (3.20*b*), when $a_3 > 0$, requires that $(\alpha_m-\beta_n)\tau < 0$ for secondary bifurcation to occur on branch σ_{mn}.

A similar analysis for the σ_{nm} branch results in the bifurcation equations

$$\left[-\frac{\tau}{\omega_0^2}(\alpha_m^2-\beta_n^2)\left\{\frac{\omega_0^2}{2\lambda_0^2}+(\lambda_0^2-\omega_0^4)\frac{\delta}{2\lambda_0^2}\right\}+a_3 b_0^2\right]A_{11} = 0, \tag{3.21a}$$

$$b_0^2 A_{12} = 0, \tag{3.21b}$$

which gives the points of secondary bifurcation on that branch. For convenience define $\epsilon_{m,n}$ to be the point of secondary bifurcation on the σ_{mn} branch and $\epsilon_{n,m}$ to be the point on the σ_{nm} branch. Then

$$\epsilon_{m,n}(\mu) = b_{m,n}\mu + O(\mu^2), \tag{3.22a}$$

$$\epsilon_{n,m}(\mu) = b_{n,m}\mu + O(\mu^2). \tag{3.22b}$$

Retaining the branch for $\epsilon > 0$ only for brevity the bifurcation equations on each branch show that

$$b_{m,n} = \left\{-\frac{\tau(\alpha_m^2-\beta_n^2)}{2\lambda_0^2 a_3(\delta)}\left[1+\frac{(\lambda_0^2-\omega_0^4)}{\omega_0^2}\delta\right]\right\}^{\frac{1}{2}}, \tag{3.23a}$$

$$b_{m,n} = \left\{\frac{\tau(\alpha_m^2-\beta_n^2)}{2\lambda_0^2 a_3(\delta)}\left[1+\frac{(\lambda_0^2-\omega_0^4)}{\omega_0^2}\delta\right]\right\}^{\frac{1}{2}}. \tag{3.23b}$$

A secondary bifurcation takes place on one, and only one at a time, branch. Noting that $\tau = \operatorname{sgn}(\xi-1)$ the branch on which the secondary bifurcation takes place is given in table 1.

$\xi-1$	$(\alpha_m-\beta_n)$	Branch
+	+	(n, m)
+	−	(m, n)
−	+	(m, n)
−	−	(n, m)

TABLE 1. The branch on which secondary bifurcation takes place

In summary, as ξ departs from $\xi = 1$, the split primary bifurcation points given by (3.5), (3.6) move away from the double point. When $\xi > 1$ they both move to the right and when $\xi < 1$ they both move to the left. However in all four cases given in the table the secondary bifurcation takes place on the branch which is emitted, after splitting, by the largest, in magnitude, of the two bifurcation points, regardless of the sign of τ.

It has been shown that in the neighbourhood of a square cross-section ($\xi = 1$) a secondary bifurcation will occur on one of the two branches emitted from the simple eigenvalues which result from the splitting of the double eigenvalues. By expanding in the neighbourhood of this point an asymptotic representation of the solution along the secondary branch may be found. A small parameter ν is defined as a measure of the distance from the point of secondary bifurcation. It is assumed that the parameters are such that the secondary bifurcation point occurs on the σ_{mn} branch. A similar analysis may be performed for a bifurcation from the σ_{nm} branch.

A perturbation is added to the known primary-branch solution

$$\phi = \epsilon\Phi + \phi', \quad \eta = \epsilon\Upsilon + \eta', \quad \omega = \sigma + \Omega, \tag{3.24a, b, c}$$

and the unknown solutions on the secondary branch are expressed as a regular perturbation series in ν and μ

$$\phi' = \nu(\mu\phi_{11}+\mu^2\phi_{12}+\ldots)+\nu^2(\mu\phi_{21}+\mu^2\phi_{22}+\ldots)+\ldots, \tag{3.25}$$

$$\eta' = \nu(\mu\eta_{11}+\mu^2\eta_{12}+\ldots)+\nu^2(\mu\eta_{21}+\mu^2\eta_{22}+\ldots)+\ldots, \tag{3.26}$$

$$\Omega = \nu(\Omega_{10}+\mu\Omega_{11}+\mu^2\Omega_{12}+\ldots)+\nu^2(\Omega_{20}+\mu\Omega_{21}+\mu^2\Omega_{22}+\ldots)+\ldots. \tag{3.27}$$

The substitution of expressions (3.24)–(3.27) into the governing equations and boundary conditions results in a set of boundary-value problems for the unknowns ϕ_{ij} and η_{ij}. The analysis although straightforward is lengthy and the details will be omitted.

The problem of first order in ν results in $\Omega_{1j} = 0$ for all j and

$$\phi_{11} = \frac{1}{\omega_0}\frac{\cosh\lambda_0(y+\delta)}{\cosh\lambda_0\delta}\cos\beta_n\bar{x}\cos\alpha_m\bar{z}\cos t, \tag{3.28a}$$

$$\eta_{11} = \cos\beta_n\bar{x}\cos\alpha_m\bar{z}\sin t, \tag{3.28b}$$

and the higher-order terms (in ν) are omitted for brevity. The problem of order ν^2 results in $\Omega_{20} = \Omega_{21} = 0$, $\phi_{21} = \eta_{21} = 0$, and

$$\begin{aligned}\frac{\Omega_{22}}{\omega_0} = {}& \frac{1}{32}\left[2\lambda_0^2 - 11\omega_0^4 - 6\frac{\alpha_m^2\beta_n^2}{\omega_0^4}\right] + \frac{[3\omega_0^4-\lambda_0^2]^2}{16\omega_0^2[4\omega_0^2-\surd 2\lambda_0\tanh(\surd 2\lambda_0\delta)]} \\ &+ \frac{[3\omega_0^4-\lambda_0^2+4\alpha_m\beta_n]^2}{32\omega_0^2[4\omega_0^2-\surd 2[\alpha_m-\beta_n)\tanh(\surd 2(\alpha_m-\beta_n)\delta)]} \\ &+ \frac{[3\omega_0^4-\lambda_0^2-4\alpha_m\beta_n]^2}{32\omega_0^2[4\omega_0^2-\surd 2(\alpha_m+\beta_n)\tanh(\surd 2(\alpha_m+\beta_n)\delta)]},\end{aligned} \tag{3.29}$$

and the other higher-order terms are omitted. The result (3.29) provides an expression for the frequency along the secondary branches. The complete expression for the natural frequency in the neighbourhood of the double eigenvalue is

$$\omega = \omega_0 + \mu^2\omega_2 + \mu^2\nu^2\Omega_{22} + O(\mu^3, \nu^3). \tag{3.30}$$

Since Ω_{22} is proportional to quadratic terms the sign of Ω_{22} determines whether the bifurcation is sub- or supercritical. Although no proof has been undertaken the following points regarding the sign $[\Omega_{22}]$ are made based on numerical evaluation of (3.29). As $\delta \to \infty$ the sign $[\Omega_{22}] < 0$ for all mode numbers. As δ is decreased a critical value of δ is reached where Ω_{22} changes sign and this critical value differs for different mode numbers. For example when $(m, n) = (1, 2)$, Ω_{22} changes sign from + to − when $\delta \sim 0.1155$ resulting in a shift of the secondary bifurcation from sub- to supercritical. Since the critical value of δ on the primary branch is slightly different from the critical value for the secondary branch there is a small range of δ where the primary branch is supercritical and the secondary branch is subcritical.

Figures 2(a, b, c, d, e) give an illustration of the secondary bifurcation phenomena for various δ when $(m, n) = (1, 2)$. Figure 1(a) for $\delta = 0.20$ is similar to the infinite-depth result obtained in Bridges (1987). The remainder of the sequence in figure 2 shows the shifting of the branches to the right as δ is decreased. In figure 2(d) the secondary bifurcation is almost vertical as $\Omega_{22} \sim 0$ here, and in figure 2(e) the secondary bifurcation has shifted to supercritical. An example of the distribution of the wave height (for $\delta \to \infty$) as the solution shifts from the primary to the secondary branch is shown in Bridges (1987). The wave field becomes more complex as the solution on the secondary branch is acquired. The addition of the finite depth is not expected to significantly alter qualitatively this distribution for suitably restricted amplitude.

Recall that the solutions discussed here are periodic free oscillations. In other words we seek the particular spatial distributions of the wave height, for example, which after an evolution in time of (normalized) period 2π will agree with the initial condition at each point in space. Therefore, physically, the secondary branches are merely additional, albeit more complex, families of solutions that satisfy this

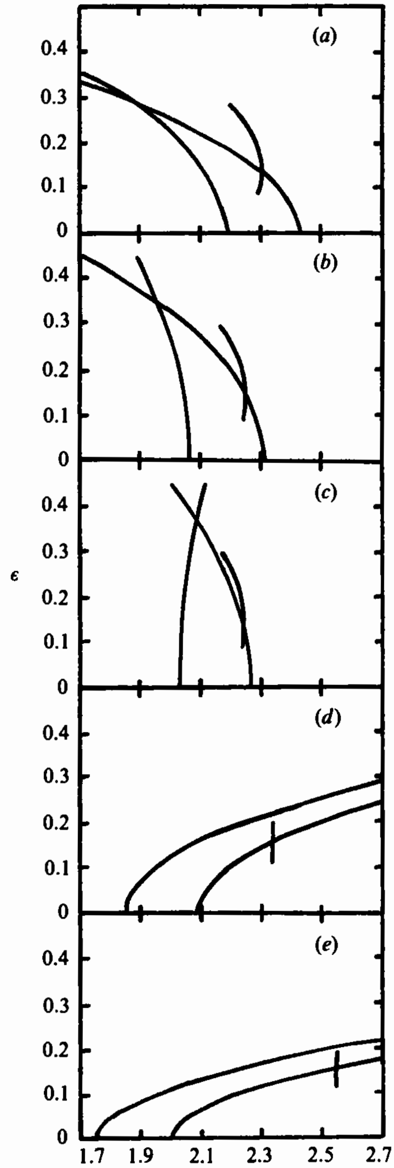

FIGURE 2. Effect of δ on the secondary bifurcation when $(m, n) = (2, 1)$ and $\xi = \frac{7}{8}$. After the splitting the secondary bifurcation occurs on the $\sigma_{2,1}$ branch. As δ is decreased the primary and then the secondary bifurcation shifts from sub- to supercritical. (*a*) $\delta = 0.20$; (*b*) 0.16; (*c*) 0.15, (*d*) 0.115; (*e*) 0.10.

property. The idea that the solutions are stable implies that the modes A_{11} and A_{12} do not exchange energy.

4. Change of type when $\epsilon = O(\delta^2)$

The solutions obtained in §§2 and 3 are not uniformly valid in the (ϵ, δ)-plane. The higher-order terms are no longer of higher order in the Boussinesq regime where

$\epsilon = O(\delta^2)$. Therefore in this section a separate analysis is performed for that region of the (ϵ, δ)-plane where $\epsilon = O(\delta^2)$. This is done by taking $\delta = (\tau\epsilon)^{\frac{1}{2}}$ where $\tau = O(1)$ and carries the sign of ϵ (and is different from the τ used in §3). The governing equations are (1.1)–(1.4) but with the scaling modified so that δ appears explicitly in the equation (y is scaled with h instead of $2a$). With this scaling the linear natural frequency has a finite non-zero limit as $\delta \to 0$.

Before proceeding to the fully three-dimensional problem it is useful to analyse weakly three-dimensional waves by (following Ablowitz & Segur 1979) considering the region of parameter space where $\xi^2 = O(\epsilon)$, or

$$\xi = (\gamma\epsilon)^{\frac{1}{2}}, \quad \delta = (\tau\epsilon)^{\frac{1}{2}}, \tag{4.1}$$

where $\gamma = O(1)$ and carries the sign of ϵ. When the relations (4.1) are substituted into the governing equations and boundary conditions and a regular expansion in ϵ is sought the leading-order problem is a wave equation

$$\mathrm{D}\psi_0(x, z, t) = 0, \tag{4.2}$$

where D is the D'Alembertian

$$\mathrm{D} \equiv \omega_0^2 \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}. \tag{4.3}$$

With the additional requirement that $\partial\psi_0/\partial n$ vanish on the vertical boundaries, (4.2) has the general solution

$$\omega_0 = \alpha_m, \tag{4.4}$$

$$\psi_0(x, z, t) = f(\zeta, z) + f(\chi, z), \tag{4.5}$$

where $\alpha_m = m\pi$, $\zeta = t + \alpha_m \bar{x}$, $\chi = t - \alpha_m \bar{x}$, and $\bar{x} = x + \frac{1}{2}$. The leading-order wave height is

$$\eta_0(x, z, t) = -\omega_0 \left[\frac{\partial}{\partial \zeta} f(\zeta, z) + \frac{\partial}{\partial \chi} f(\chi, z)\right]; \tag{4.6}$$

the unknown function f is found through application of the solvability condition at the next order. The first-order problem is

$$\mathrm{D}\psi_1 = F_1(x, z, t), \tag{4.7}$$

where F_1 is a functional of zeroth-order terms. Solvability of (4.7) requires that

$$\int_0^{2\alpha_m} F_1\left[\frac{\zeta - \chi}{2\alpha_m}, z, \frac{\zeta + \chi}{2}\right] \mathrm{d}\zeta = 0. \tag{4.8}$$

This condition is derived in Bridges (1986). Application of (4.8) results in a partial differential equation for f:

$$\tau \frac{\partial^4 f}{\partial \chi^4} - \frac{6\omega_1}{\omega_0^3} \frac{\partial^2 f}{\partial \chi^2} - \frac{9}{\omega_0} \frac{\partial f}{\partial \chi} \frac{\partial^2 f}{\partial \chi^2} + 3\frac{\gamma}{\omega_0^4} \frac{\partial^2 f}{\partial z^2} = 0. \tag{4.9}$$

When $\gamma = 0$ the equation for f can be integrated to yield

$$f'(\chi) = A + B\,\mathrm{cn}^2(\chi; \kappa) \tag{4.10}$$

(where A and B are constants), the usual cnoidal wave, and subsequent substitution into (4.5) results in a standing cnoidal wave (Bridges 1986). With the retention of the $\partial^2 f/\partial z^2$ term (4.10) is a form of the K–P equation. The K–P equation was first derived by Kadomtsev & Petviashvili (1970) in their study of the stability of solitary waves to transverse perturbations. A discussion and analysis of this equation can be found in Ablowitz & Segur (1979). Dubrovin (1981) and Segur & Finkel (1985) have shown that this equation is rich in the number of qualitatively different types of solutions that may be produced. Here the right and left running solutions of the K–P equation would be combined to form a weakly three-dimensional standing wave.

Instead of analysing this equation and its possibilities further an analysis with ξ unrestricted will be undertaken. With ξ unrestricted and $\delta = (\tau\epsilon)^{\frac{1}{2}}$ a regular perturbation expansion in ϵ is assumed. Substitution into the governing equations and boundary conditions results in a wave equation in two space dimensions at leading order:

$$\omega_0^2 \frac{\partial^2 \psi_0}{\partial t^2} - \Delta\psi_0 = 0, \tag{4.11}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \xi^2 \frac{\partial^2}{\partial z^2}, \tag{4.12}$$

and it is required that the normal derivative of ψ_0 vanish at the vertical boundaries. The leading-order wave height is given by

$$\eta_0(x, z, t) = -\omega_0 \frac{\partial}{\partial t} \psi_0(x, z, t). \tag{4.13}$$

The first-order problem results in

$$\omega_0^2 \frac{\partial^2 \psi_1}{\partial t^2} - \Delta\psi_1 = \frac{\omega_0^4}{3} \frac{\partial}{\partial t} G_1(x, z, t), \tag{4.14}$$

where

$$G_1(x, z, t) = -\frac{3}{\omega_0^3}\left[\left(\frac{\partial \psi_0}{\partial x}\right)^2 + \xi^2\left(\frac{\partial \psi_0}{\partial z}\right)^2 + \tfrac{1}{2}\omega_0^2\left(\frac{\partial \psi_0}{\partial t}\right)^2\right] + \tau \frac{\partial^3 \psi_0}{\partial t^3} - \frac{6\omega_1}{\omega_0^3}\frac{\partial \psi_0}{\partial t}. \tag{4.15}$$

For solvability it is required that

$$\int_0^{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \psi_0}{\partial t} G_1(x, z, t)\, \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}t = 0. \tag{4.16}$$

The function $\psi_0(x, z, t)$ which satisfies this functional differential equation is the leading-order term for three-dimensional standing waves in a rectangular basin when $\epsilon = O(\delta^2)$. A general solution to (4.16) has not been found. However it may be shown by substitution that with

$$\omega_1 = -2\tau\omega_0^3\left[\frac{\mathrm{E}(\pi;\kappa)}{\pi} - \tfrac{1}{3}(2-\kappa^2)\right], \tag{4.17}$$

one possible expression for the leading-order wave height which satisfies (4.16) is

$$\eta_0(x, z, t) = h(t + \alpha_m \bar{x} + \beta_n \bar{z}) + h(t - \alpha_m \bar{x} - \beta_n \bar{z}) + h(t - \alpha_m \bar{x} + \beta_n \bar{z}) + h(t + \alpha_m \bar{x} - \beta_n \bar{z}), \tag{4.18}$$

where

$$h(\rho) = A + B\, \mathrm{cn}^2(\rho;\kappa), \tag{4.19}$$

and

$$A = -\tfrac{4}{3}\omega_0^2 \tau \left\{\kappa^2 - 1 + \frac{\mathrm{E}(\pi;\kappa)}{\pi}\right\}, \tag{4.20}$$

$$B = \tfrac{4}{3}\kappa^2 \tau \omega_0^2 \tag{4.21}$$

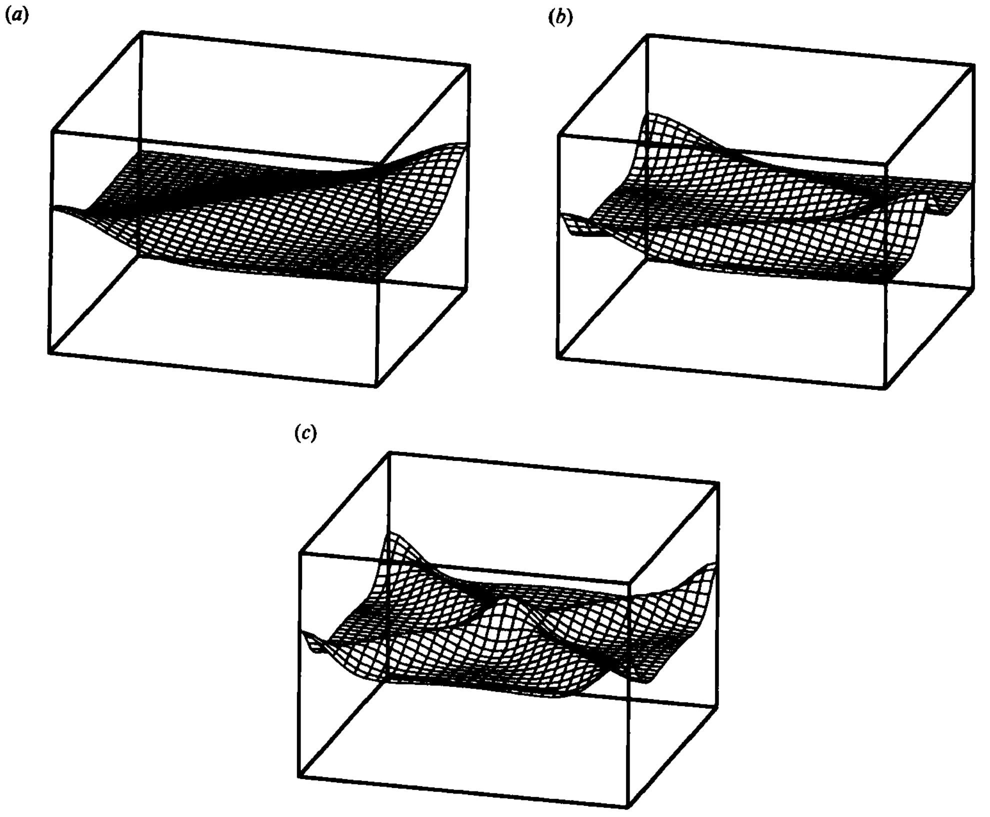

FIGURE 3. A three-dimensional cnoidal standing wave for (a) $(m, n) = (1, 1)$; (b) $(1, 2)$; (c) $(2, 2)$, $t = 0$.

where $\mathrm{E}(\pi;\kappa)$, and $\mathrm{cn}\,(\rho;\kappa)$ are Jacobian elliptic functions (Byrd & Friedman 1971). Periodicity in time and the finite domain require that κ^2, the modulus of the elliptic functions, satisfy the equation

$$\pi-\int_0^{\frac{1}{2}\pi}\frac{\mathrm{d}\zeta}{(1-\kappa^2\sin^2\zeta)^{\frac{1}{2}}}=0. \tag{4.22}$$

A numerical evaluation shows that $\kappa^2 \sim 0.9691$.

Therefore a solution of (4.16) is a set of four oblique, travelling, non-interacting (to leading order) cnoidal waves which when combined result in a nonlinear three-dimensional standing cnoidal wave. It is also illuminating to note that the wave height (4.18) may be expressed as the infinite sum

$$\eta_0(x, z, t) = k_0 \sum_{p=1}^{\infty} p a_p \cos p\alpha_m \bar{x} \cos p\beta_n \bar{z} \cos pt, \tag{4.23}$$

where k_0 is a constant and

$$a_0 = \frac{1}{\kappa^2}\left[\frac{\mathrm{E}(\pi;\kappa)}{\pi}+\kappa^2-1\right], \tag{4.24a}$$

$$a_p = \frac{2}{\kappa^2}\frac{pq^p}{1-q^{2p}} \quad \text{for } p>0, \tag{4.24b}$$

and $q = \exp\left[-K(\tfrac{1}{2}\pi;(1-\kappa^2)^{\frac{1}{2}})\right]$.

Examples of the three-dimensional cnoidal standing waves are given in figure 3(*a*, *b*, *c*). Figure 3(*a*) is a $(m, n) = (1, 1)$ mode at $t = 0$, figure 3(*b*) is a $(m, n) = (1, 2)$ mode at $t = 0$, and figure 3(*c*) is a $(m, n) = (2, 2)$ mode at $t = 0$. These figures give a prelude to the richness that is possible when a complete solution of (4.16) is found.

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